

# Classicality as a Property of Predicate Symbols

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## Introduction

Investigation of the space between intuitionistic and classical logics has been focused on intermediate (superintuitionistic) logics. They are defined by adding one or more axiom schemas weaker than the law of excluded middle to intuitionistic logic [CZ]. For example, the Kreisel-Putnam logic is defined by adding the axiom schema:  $(\neg A \supset B \vee C) \supset (\neg A \supset B) \vee (\neg A \supset C)$ .

Yet another approach to looking into the space between intuitionistic and classical logics is to view classicality as a property of individual predicate symbols. Let us call predicate symbol  $P$  classical (or decidable) if all  $P(a_1, \dots, a_n)$  are decidable, i.e.  $P(a_1, \dots, a_n) \vee \neg P(a_1, \dots, a_n)$  holds for any choice of  $a_1, \dots, a_n$ . Consider extensions of intuitionistic logic defined by sets of classical predicate symbols. This approach was originally proposed by Sergei Artemov. Hidenori Kurokawa researched the propositional case [Ku]. The first-order case where all propositional symbols are classical whereas none of predicate symbols of arity one or more is classical is investigated in [Sa].

All these extensions are partially ordered and form a lattice. The order is induced by the subset relation for sets of classical predicate symbols. Union and intersection for these sets define the lattice. Intuitionistic logic is the lower bound of the lattice and classical logic is the upper bound. All the extensions are consistent. It will be shown later that there is a straightforward translation between the extensions for ordered pairs of sets of classical predicate symbols. These extensions could be combined with intermediate logics too.

We will use Gentzen's calculus LJ as a framework [Ge]. In sequent notation, assumption that predicate symbol  $P$  is classical (decidable) is expressed as axiom schema  $\vdash P(a_1, \dots, a_n) \vee \neg P(a_1, \dots, a_n)$  applicable to any terms  $a_1, \dots, a_n$ . Alternatively, decidability of predicate symbols can be expressed as Rule of Excluded Middle (REM) applicable to  $N$  containing classical predicate symbols only [NP]:

The two definitions are equivalent, that is, the sets of derivable formulas are the same. This equivalence is proved by replacing REMs with  $\vdash N \vee \neg N$  and vice versa. For any set  $D$  of decidable predicate symbols, let  $L(D)$  be intuitionistic logic extended with REM for formulas whose all predicate symbols are from  $D$ .

For any non-empty  $D$ , there exist formulas provable in  $L(D)$  but not provable intuitionistically. If  $R$  belongs to  $D$ , then formula  $R(a_1, \dots, a_n) \vee \neg R(a_1, \dots, a_n)$  is such. Similarly, if there is a predicate symbol not belonging to  $D$ , then there exist formulas provable classically but not in  $L(D)$ . Also, if  $D$  is not a subset of  $E$ , then there exist formulas provable in  $L(D)$  but not in  $L(E)$ . Every  $L(D)$  is closed under modus ponens and under substitution of formulas for predicate symbols with the same variables free, with the restriction that only formulas whose symbols are all from  $D$  can be substituted for symbols

from D.

We will extensively use the well-known notions of positive, negative and strictly positive subformula occurrences. A predicate symbol will be called unipolar in a given formula if its atoms in this formula are either all positive or all negative. Otherwise, it is called bipolar. A formula is called D-unipolar if all symbols belonging to D are unipolar in it.

For any choice of D, the set of formulas derivable in  $L(D)$  is undecidable due to the fact that any formula is classically derivable if and only if its double negation translation is derivable in  $L(D)$ . The same argument can be used to prove undecidability of other fragments of  $L(D)$  for which double negation translations belong to the same fragment. For instance, the three-variable fragment of  $L(D)$  is undecidable for any D because it is undecidable in classical logic. If for any k there exist predicate symbols of arity k not belonging to D, and if a fragment of  $L(D)$  is closed under substitution among predicate symbols, then decidability in  $L(D)$  agrees with that in intuitionistic logic.

## Proof-Theoretical Preliminaries

**Theorem** (weak cut elimination). Every  $L(D)$  derivation for any D can be transformed into another derivation with the same endsequent and in which the cut rule does not occur.

This theorem is proved by using Gentzen's cut elimination proof for LJ [Ge] and augmenting it with two transformations that swap REM and the mix rule. One of the transformations lowers the left rank, the other lowers the right rank. REM is another discharge rule along with the cut rule, and it cannot be eliminated from  $L(D)$  derivations for non-empty D. Negri and von Plato proved admissibility of cut in  $G3ip$  extended with REM [NP].

**Theorem** (weak subformula property). Every derivation in  $L(D)$  for any D can be transformed into such derivation with the same endsequent that all subformulas from this derivation occur in the endsequent except for negations of the atoms whose symbols belong to D and that are bipolar in the endsequent.

To prove this theorem, we consider a derivation with using axioms  $\vdash A \vee \neg A$  applied to atoms only. This derivation is turned into a derivation with REM, and cut is eliminated. REMs still apply exclusively to atoms. For any predicate symbol N that either does not occur in the endsequent or is unipolar in it, there exist a topmost REM in which one of the upper sequents contains occurrences of  $N(\dots)$  of one polarity only. Due to the sign property [KI],  $N(\dots)$  can be introduced in the derivation part above this sequent by thinnings only. Therefore, such rules can be eliminated. This theorem gives a somewhat stronger subformula property than the variant from [NP]. Note that by this theorem,  $L(D)$  coincides with intuitionistic logic on the class of formulas not containing bipolar symbols from D.

## Classical Features

The following theorem is characteristic of classical logic, and it does not hold in intuitionistic logic. If R is a propositional symbol occurring in formula A, then let  $A\{R|F\}$  denote the formula obtained from A by replacing all occurrences of R with formula F.

**Theorem.** If R is a propositional symbol from D and occurring in formula A, then A is derivable in  $L(D)$  if and only if both  $A\{R|\top\}$  and  $A\{R|\perp\}$  are derivable in  $L(D)$ .

The ‘only if’ part is proved by replacing the axioms  $\vdash \top, \perp \vdash$  with  $R \vdash R$  in derivations of  $A\{R|\top\}$  and  $A\{R|\perp\}$ , adjusting the rest of the derivations, and using REM for R at the bottom. The ‘if’ part follows from the replacement theorem.

Kleene compiled an extensive collection of logical laws [Kl]. Most laws from this collection hold in intuitionistic logic but still some are classical-only. Many of these laws have two components (A and B). Below are those classical-only laws from the Kleene collection which also hold in  $L(D)$  provided that one of the two components contain predicate symbols from D only.

- $A \supset B \sim \neg B \supset \neg A$  - holds if all symbols from B belong to D
- $\neg(A \& B) \sim \neg A \vee \neg B$  - holds if all symbols from A or all symbols from B belong to D
- $\neg(A \supset B) \sim A \& \neg B$  - holds if all symbols from A belong to D
- $A \supset B \sim \neg A \vee B$  - holds if all symbols from A or all symbols from B belong to D
- $A \vee B \sim \neg A \supset B$  - holds if all symbols from A belong to D
- $A \vee \forall x B(x) \sim \forall x (A \vee B(x))$  - holds if x does not occur free in A and all symbols from A belong to D
- $A \supset \exists x B(x) \sim \exists x (A \supset B(x))$  - holds if x does not occur free in A and all symbols from A belong to D

Beyond propositional formulas,  $L(D)$  coincides with classical logic on an extension of the class of Harrop formulas. This extension  $H(D)$  is defined recursively by the following:

- $\top, \perp, \neg A, B$  are  $H(D)$  formulas whenever all predicate symbols from B belong to D
- $A \& B$  is a  $H(D)$  formula whenever A and B are  $H(D)$  formulas
- $A \vee B$  is a  $H(D)$  formula whenever all predicate symbols from one of A, B belong to D and the other is a  $H(D)$  formula
- $A \supset B$  is a  $H(D)$  formula whenever B is a  $H(D)$  formula
- $\forall x A(x)$  is a  $H(D)$  formula whenever  $A(x)$  is a  $H(D)$  formula

The fact that every classically derivable  $H(D)$  formula is also derivable in  $L(D)$  is established by proving that  $H(D)$  formulas are stable and equivalent to their double negation translations in  $L(D)$ . We leverage on the proof of the same property of Harrop formulas in intuitionistic logic. The two additional cases to cover are: formulas whose all predicate symbols are from D; disjunctions.

## Derivability Correspondence

**Definition.** Subformula occurrence G is called a block component of formula F in  $L(D)$  if every subformula occurrence above G (i.e. having G as a proper part) is one of the following:

- $\neg A$ , strictly positive  $\forall x A(x)$ , strictly positive  $\exists x A(x)$ , where A is not G
- positive  $A \& B$ , or negative  $A \vee B$ , or negative  $A \supset B$ ; if  $A \& B / A \vee B / A \supset B$  is in the scope of  $\exists x$ , then one of A, B does not contain x free
- positive  $A \vee B$ , or positive  $A \supset B$ , or negative  $A \& B$ , one of A, B contains symbols from D only and if  $A \vee B / A \supset B / A \& B$  is in the scope of  $\forall x$ , then it does not contain x free, every symbol from both A and B, and belonging to D is unipolar in  $A \vee B / A \supset B / A \& B$

**Theorem.** If F is derivable in  $L(D)$  and its every minimal block component, i.e. not containing other block components as its parts, is either a formula whose every symbol from D-E is unipolar in it or a Harrop/non-strictly positive formula not containing x free if it is in the scope of  $\exists x$ , then F is also derivable in  $L(E)$ .

**Corollary.** If F is derivable in  $L(D)$  and its every minimal block component is either a D-unipolar

formula or a Harrop/non-strictly positive formula not containing  $x$  free if it is in the scope of  $\exists x$ , then  $F$  is also derivable intuitionistically.

This theorem is first proved for the class of formulas in which only connectives occurring above any of minimal block components  $F_1, \dots, F_n$  are  $\&$  and  $\vee$ , no quantifier occurs above any of  $F_1, \dots, F_n$ . The proof is by induction on the minimal length of all paths from the root of the tree representing  $F$  to the roots of  $F_1, \dots, F_n$ . The base case and the induction step for the case when  $F$  is  $A\&B$  are straightforward. The induction step for the case when  $F$  is  $A\vee B$  is proved by using the disjunction property theorem from the next section.

After that, it is proved that any formula satisfying the premise of this theorem can be converted to a formula from the aforementioned class by a sequence of equivalent transformations. These transformations eliminate all implications above  $F_1, \dots, F_n$ . Negations and quantifiers are pushed down to  $F_1, \dots, F_n$ . Non-strictly positive block components become negated.

This theorem could be strengthened by allowing not only Harrop but also other classes of formulas for which classical and intuitionistic derivability coincide [Na, Or]. This would require, though, imposing additional conditions on polarity and presence of variables bound outside of minimal block components. Note that derivability in both  $L(D)$  and  $L(E)$  does not imply derivability in  $L(D\cap E)$ .

## Disjunction and Existence Properties

Obviously, the disjunction property does not hold for the entire  $L(D)$  for any non-empty  $D$ . The existence property does not hold in  $L(D)$  with non-empty  $D$  either. The following two theorems give an insight on the extent of the disjunction and existence properties.

**Theorem.** If formula  $F\vee G$  is derivable in  $L(D)$ , and every symbol from  $D$  and occurring in both  $F$  and  $G$  is unipolar in  $F\vee G$ , then either  $F$  or  $G$  is derivable in  $L(D)$ .

The weak subformula property guarantees that there is a derivation of  $F\vee G$  without cut and in which REMs apply only to atoms whose symbols do not occur in one of  $F, G$ . REMs below complementary  $\vee$ -succedent rules resulting in  $F\vee G$  are eliminated. Thus, we get a derivation of either  $F$  or  $G$ . This theorem strengthens Kurokawa's disjunction property result [Ku].

Here is a counterexample showing that the existence property does not hold in  $L(D)$  with non-empty  $D$ . Let  $A$  be any unary predicate symbol and  $N$  be a unary predicate symbol from  $D$ . (The case of another arity would be similar.) Formula  $\exists x ((A(a)\vee N(d))\&(A(b)\vee\neg N(d)) \supset A(x))$  is derivable in  $L(D)$ . For any term replacing  $x$ , the respective formula is not even classically derivable. It is shown by building a falsifying classical model.

**Theorem.** The existence property holds for  $\exists x F(x)$  derivable in  $L(D)$  if its every minimal block component is either a  $D$ -unipolar formula or a formula not containing  $x$ .

Note that this class of formulas includes the formulas from the corollary in section Derivability Correspondence. Proof of this theorem is similar to the proof of the theorem in that section.

## Translations

It is clear from the proof of the weak subformula property that formula  $F$  is derivable in  $L(D)$  if and only if  $A \supset F$  is derivable intuitionistically; where  $A$  is conjunction of  $\forall a_1 \dots \forall a_n (R(a_1, \dots, a_n) \vee \neg R(a_1, \dots, a_n))$  for all symbols  $R \in D$  that are bipolar in  $F$ . This relationship can be viewed as translation of  $L(D)$  into intuitionistic logic. If  $D$  is a subset of  $E$ , then we can similarly translate  $L(E)$  into  $L(D)$ . Formula  $F$  is derivable in  $L(E)$  if and only if  $A \supset F$  is derivable in  $L(D)$ ; where  $A$  is a similar conjunction for all symbols  $R \in E - D$  that are bipolar in  $F$ . Furthermore, for any  $D$  and  $E$ , if formula  $F$  is derivable in  $L(E)$ , then  $A \supset F$  is derivable in  $L(D)$ ; where  $A$  is same as above.

As previously mentioned, double negation translation works for classical logic and  $L(D)$  for any  $D$ . Similarly, formula  $F$  is classically provable if and only if  $\neg \neg F$  is provable in  $L(D)$  extended with the double negation shift. In application to classical logic and  $L(D)$ , double negation translation can be made less 'intrusive'. First, formulas whose all symbols belong to  $D$  do not have to be negated. Second, disjunctions having at least one component with all symbols from  $D$  do not have to be translated either. The usual proof connecting classical and intuitionistic derivability for double negation translation applies to the less intrusive translation since the two aforementioned classes of formulas are double negation closed in  $L(D)$ .

If  $D$  contains propositional symbols only, then  $F$  is derivable in  $L(D)$  if and only if all  $2^n$  formulas of the form  $F\{R_1|P_1\} \dots \{R_n|P_n\}$  are derivable intuitionistically. Here,  $R_1 \dots R_n$  are all distinct bipolar propositional symbols from  $D$  and  $P_1 \dots P_n$  range over all distinct sequences of  $\top$  and  $\perp$ . This is proved by using the theorem from section Classical Features. If  $E$  is a subset of  $D$  and  $D - E$  contains propositional symbols only, then formula  $F$  is derivable in  $L(D)$  if and only if conjunction of  $2^n$  formulas  $F\{R_1|P_1\} \dots \{R_n|P_n\}$  is derivable in  $L(E)$  where  $R_1 \dots R_n$  are all distinct bipolar propositional symbols from  $D - E$  and  $P_k$  are same as above. These relationships can be viewed as translations as well.

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